Projection Matrix

1. Definition: A projection matrix is a matrix that when multiplied by a vector gives the projection of the vector onto the matrix's subspace.

   Note: Projections of vectors in R^n on a subspace, W, gives a mapping T of R^n into itself. T is a linear transformation. Since T is a linear transformation, there must be a matrix P, s.t. $T(x) = P(x)$. This shows that P is the standard matrix representation of T and is the projection matrix.

2. Thm: Let A be a $m \times n$ matrix of rank. Then, the $n \times n$ matrix A^TA also has rank r.

Proof:

Let $x \in \text{Nullspace}(A)$

Then,

$Ax = 0$

$A^TAx = 0$

$x \in \text{Nullspace}(A^TA)$

Hence, $\text{Nullspace}(A) \subseteq \text{Nullspace}(A^TA)$.

Now, let $x \in \text{Nullspace}(A^TA)$

$A^TAx = 0$

$x^TA^TAx = 0$

$(Ax)^TAx = 0$

$Ax = 0$

$x \in \text{Nullspace}(A)$

Hence, $\text{Nullspace}(A^TA) \subseteq \text{Nullspace}(A)$

Since nullspace(A) = nullspace(A^TA),
dim(ns(A)) = dim(ns(A^TA))
Since both A and A^TA has n columns,
rank(A) = rank(A^TA)

QED

3. Formula:

Let $w = sp(a_1, a_2, \dots a_k)$ be a k-dimensional subspace of R^n, and let A have as col vectors $a_1, a_2, \dots a_k$. Then, the projection of b in R^n on w is given by the formula $b_w = A(A^TA)^{-1}A^Tb$. Note, $A(A^TA)^{-1}A^T$ is the projection matrix.

4. Examples:

1. Find the projection of b on $w = sp(a)$ given

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } a = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

Solution:

$$a^T = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix}$$

$$a^Ta = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 29 \end{bmatrix}$$

$$(a^Ta)^{-1} = \left(\frac{1}{29}\right)$$

$$b_w = a(a^Ta)^{-1}a^Tb$$

$$= \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}\left(\frac{1}{29}\right)\begin{bmatrix} 2 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \left(\frac{1}{29}\right) \begin{bmatrix} 4 & 8 & 6 \\ 8 & 16 & 12 \\ 6 & 12 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \left(\frac{1}{29}\right) \begin{bmatrix} 38 \\ 76 \\ 57 \end{bmatrix}$$

$$= \left(\frac{19}{29}\right) \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

2. Find the projection matrix that projects vectors in R^3 onto the plane $2x-y-3z=0$.

Since the plane contains the zero vector, it can be written as the subspace $W = sp(a_1, a_2)$ where a_1 and a_2 are non-zero and non-parallel vectors.

let $a_1 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 10 & 6 \\ 6 & 5 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{14} \begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}$$

$\therefore$ The projection matrix is

$$\frac{1}{14}\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ -6 & 10 \end{bmatrix}\begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{bmatrix}$$

5. Properties of the Projection Matrix (P):

1. $P^2 = P$ (P is idempotent)
2. $P^T = P$ (P is symmetric)

6. The Orthonormal Case:

Let $\{a_1, a_2, \dots a_k\}$ be an orthonormal basis for the subspace W of $\mathbb{R}^n$. Then, $P = AA^T$, where A is the $n \times k$ matrix having as col vectors $a_1, a_2, \dots a_k$.